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**Competitive equilibrium with asymmetric
information : an existence theorem
for numeraire assets**

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COMPETITIVE EQUILIBRIUM WITH ASYMMETRIC INFORMATION: AN EXISTENCE THEOREM FOR NUMERAIRE ASSETS

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Abstract

In a general equilibrium model of incomplete markets with nominal assets and adverse selection, Cornet-De Boisdeffre [3] introduced refined concepts of “no-arbitrage” prices and equilibria, which extended to the asymmetric information setting the classical concepts of symmetric information. We now present the model with numeraire assets and study its existence properties. We show that equilibrium exists, as long as financial markets preclude arbitrage, under similar standard conditions, whether agents have symmetric or asymmetric information. This result departs from the rational expectations’ outcome and extends to the asymmetric setting the classical existence property of symmetric information models with numeraire assets.

Key words: general equilibrium, asymmetric information, arbitrage, inference, existence of equilibrium.

JEL Classification: D52.

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1 Introduction

When agents are asymmetrically informed on financial markets, the existence of equilibrium, as well as the level of equilibrium prices and allocations, depend crucially on how agents update their beliefs from observing prices or volumes traded on markets. The issue, however, is debated: *“the theory with asymmetric information, Ross argues [12, p. 8], is not well understood at all. In short, the exact mechanism by which prices incorporate information is still a mystery and an attendant theory of volume is simply missing.”* The traditional response to that problem is given by the R.E.E. (rational expectations equilibrium) models of asymmetric information, by assuming, quoting Radner, 1979 [11], that *“agents have a ‘model’ or ‘expectations’ of how equilibrium prices are determined”*. Our approach, based on arbitrage, does not use this assumption: agents having asymmetric information and no price expectations update their beliefs by analysing arbitrage opportunities on financial markets. In [4], we present this “no-arbitrage principle” and show how it leads agents to infer in a decentralized manner a correct information from prices, which precludes arbitrage. In this approach, the classical financial equilibrium and equilibrium with asymmetric information become two sides of a same broad concept.

This refined concept of equilibrium, and a related notion of no-arbitrage price, were introduced in Cornet-De Boisdeffre, 2002, [3] in a two-period pure exchange financial economy, with incomplete nominal-asset markets and asymmetric information of the adverse selection’s type: the simplest model to analyse arbitrage. Such concepts of price and equilibrium were shown to coincide with the classical ones, under symmetric information, and to extend them to the asymmetric information setting.

We now present the model with asymmetric information and numeraire assets, the corresponding concepts of price and equilibrium, and the existence properties of equilibrium. Departing from the rational expectations’ generic existence outcome [11], we prove that equilibrium exists, in our model, whenever financial markets preclude arbitrage, that is, the minimum requirement for existence is also a sufficient condition. In a companion paper, we show the same property holds for nominal assets. This result is satisfied under similarly mild conditions with symmetric and asymmetric information and allows any pattern of asymmetry at equilibrium. It extends the classical existence theorem for numeraire assets demonstrated by Geanakoplos-Polemarchakis, 1986 [8], in the symmetric information case. The proof is specifically tailored to apply to the broadest informational pattern, and simplifies under symmetric information.

Section 2 recalls the main characteristics and definitions of the model: information structures and refinements, markets, consumers’ behavior and the concepts of no-arbitrage price and equilibrium. Section 3 states and proves the existence theorem along the Gale-Mas-Colell [6, 7] fixed-point-like argument. An Appendix proves technical Lemmas.

2 The basic model

We consider a pure-exchange financial economy with two periods ($t \in \{0, 1\}$). The economy is finite, in the sense that the sets of agents, $I := \{1, \dots, m\}$, of commodities, $\{1, \dots, L\}$, of states of nature, S , and of assets, $\{1, \dots, J\}$, are finite. There is an a priori uncertainty at the first period ($t = 0$) about which state $s \in S$ will prevail at the second period ($t = 1$). We shall denote by $s = 0$ the non-random state at $t = 0$ and let $\Sigma' := \{0\} \cup \Sigma$, for any subset Σ of S .

2.1 Information structures and refinements

At the first period, each agent $i \in I$ receives a private signal, or “information set” $S_i \subset S$, which informs the agent that an arbitrary state $s \in S_i$ will prevail at $t = 1$ (e.g., $S_i = S$ is the non-informative signal). Agents may then update their beliefs by exchanging or inferring information, e.g., from observing prices. But we assume this information is always correct, in the sense that each information set contains the “true” state, which will prevail at $t = 1$. This specification departs from the notion of unawareness, introduced by Modica et alii [10], where agents ignore some states, which may ultimately prevail. It yields the following definitions of information structures and refinements.²

Definition 1 *A collection (Σ_i) of m subsets of S is called an information structure, or structure, if $\cap_{i=1}^m \Sigma_i \neq \emptyset$. A structure is said to be symmetric if all its information sets are equal. The set of all structures is denoted by Γ . Let $(\Sigma_i) \in \Gamma$ be given. We denote by $\underline{\Sigma}$ the pooled information set $\cap_{i=1}^m \Sigma_i$. We say that a structure $(\tilde{\Sigma}_i)$ is a refinement of (Σ_i) , or refines (Σ_i) , and denote it $(\tilde{\Sigma}_i) \leq (\Sigma_i)$, if $\underline{\Sigma} \subset \tilde{\Sigma}_i \subset \Sigma_i$ for each $i \in I$. We let $(\underline{\Sigma}_i)$ be the symmetric refinement of (Σ_i) . Given a $S \times J$ -matrix V , the couple $[V, (\Sigma_i)]$ is also referred to as a structure.*

2.2 Markets and consumers’ behavior

The economy comprises a financial market and a commodity market.

The financial market permits limited transfers across periods and states, via J numeraire assets $j \in \{1, \dots, J\}$, whose contingent payoffs, in each state $s \in S$, are denoted by $v_j[s]e$, where $e \in \mathbb{R}_+^L \setminus \{0\}$ is the numeraire, that is, a fixed bundle of commodities (and we let $\|e\| = 1$ for simplicity), and $v_j[s]$ is a state-dependent quantity. These quantities, defined for each $(s, j) \in S \times \{1, \dots, J\}$, yield a $S \times J$ -matrix $V := (v_j[s])$, which is of full column-rank (i.e., $J = \text{rank} V$).³

²The following definition of a refinement corresponds to the restrictive notion of “self-attainable” refinement in [3], with reference to which equilibrium will be defined.

³Throughout the paper, the scalar product and Euclidean norm are denoted by \cdot and $\|\cdot\|$, respectively. For each $\Sigma \subset S'$, every $S \times J$ -matrix $V := (v_j[s])_{(s,j) \in S \times J}$ and $\Sigma \times J$ -matrix A , for all vector collections $(x, x', y, y', z, z') \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^{S'} \times \mathbb{R}^\Sigma \times \mathbb{R}^\Sigma \times (\mathbb{R}^L)^\Sigma \times (\mathbb{R}^L)^\Sigma$ and $(p, q, s, l) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J \times \Sigma \times \{1, \dots, L\}$, we denote by:

1) $x[\Sigma]$ and $x'[\Sigma]$, respectively, the truncations of x on $(\mathbb{R}^L)^\Sigma$ and of x' on \mathbb{R}^Σ ;

Thus, for every price $p \in (\mathbb{R}^L)^{S'}$, the real numbers $(p[s] \cdot e)v_j[s]$, for each $(s, j) \in S \times \{1, \dots, J\}$, define a $S \times J$ price-dependent payoff matrix in units of account, $V(p) := ((p[s] \cdot e)v_j[s])$, which is of full column-rank whenever $p[s] \cdot e > 0$ for each $s \in S$. Given the asset price $q \in \mathbb{R}^J$, a portfolio is a vector $z \in \mathbb{R}^J$, tradable for $q \cdot z$ units of account at $t = 0$, which promises delivery of a flow Vz of contingent payoffs in numeraire at $t = 1$.

There are also $\#S'$ spot markets for each commodity $l \in \{1, \dots, L\}$, which may be traded or consumed at both dates. For each $\Sigma \subset S$, the generic agent $i \in I$ has, conditionally on the information Σ she may attain at $t = 0$, a consumption set, $X(\Sigma) := \{x \in (\mathbb{R}_+^L)^{S'} : x[s] = 0 \text{ for every } s \in S \setminus \Sigma\}$, an endowment, $e_i(\Sigma) \in X(\Sigma)$, and a preference correspondence P_i^Σ represented by a utility function, $u_i^\Sigma : X(\Sigma) \rightarrow \mathbb{R}$, and defined, for every $x \in X(\Sigma)$, by $P_i^\Sigma(x) := \{y \in X(\Sigma) : u_i^\Sigma(y) > u_i^\Sigma(x)\}$.

For every $i \in I$ and every $\Sigma \subset S$, we assume that u_i^Σ is separable, in the sense that there exist utility indexes, $v_i(\Sigma, s) : \mathbb{R}_+^2 \rightarrow \mathbb{R}$, for each $s \in \Sigma$, such that $u_i^\Sigma(x) := \sum_{s \in \Sigma} v_i(\Sigma, s)(x[0], x[s])$, for every $x \in X(\Sigma)$. This specification, which encompasses the case of V.N.M. utility functions, will permit to set a strictly positive bound to the value of the numeraire on every spot market (see the Appendix), as required to bound strategies. Using the notations in footnote, we shall refer, throughout, to the following Assumptions, which are standard (e.g., see Geanakoplos-Polemarchakis [8] in the symmetric no-information case) and given for every $i \in I$, and every $\Sigma \subset S$.

Assumption A1 (non satiation in the numeraire in any state):

$$\forall (x, s) \in X(\Sigma) \times \Sigma', u_i^\Sigma(x + e_s) > u_i^\Sigma(x), \text{ where } e_s[s] := e, e_s[S' \setminus \{s\}] := 0.$$

Assumption A2 (strong survival): $\forall s \in \Sigma', e_i(\Sigma)[s] > 0$.

Assumption A3 (continuity):

$$\forall \varepsilon > 0, \forall x \in X(\Sigma), \exists \eta > 0, \text{ s.t. } \tilde{x} \in X(\Sigma), \|\tilde{x} - x\| < \eta \implies |u_i^\Sigma(\tilde{x}) - u_i^\Sigma(x)| < \varepsilon.$$

Assumption A4 (quasi-concavity): $u_i^\Sigma(x + \lambda(y - x)) \geq \min(u_i^\Sigma(x), u_i^\Sigma(y))$, for each $((x, y), \lambda) \in X(\Sigma)^2 \times]0, 1[$, with a strict inequality if $u_i^\Sigma(x) \neq u_i^\Sigma(y)$.

-
- 2) $A[s]$, $y[s]$, $z[s]$, respectively, the row, scalar and vector, indexed by $s \in \Sigma$, of A , y , z ;
 - 3) $z^l[s]$ the l^{th} component of $z[s] \in \mathbb{R}^L$ and $z^l := (z^l[s]) \in \mathbb{R}^\Sigma$;
 - 4) $y \leq y'$ and $z \leq z'$ (resp. $y << y'$ and $z << z'$) the relationships $y[s] \leq y'[s]$ and $z^l[s] \leq z'^l[s]$ (resp. $y[s] < y'[s]$ and $z^l[s] < z'^l[s]$) for every $s \in \Sigma, l \in \{1, \dots, L\}$;
 - 5) $y < y'$ (resp. $z < z'$) the joint relationships $y \leq y'$ and $y \neq y'$ (resp. $z \leq z'$ and $z \neq z'$);
 - 6) $z_\square z'$ the vector $(z[s] \cdot z'[s]) \in \mathbb{R}^\Sigma$, $y_\square z$ the vector $(y[s]z[s]) \in (\mathbb{R}^L)^\Sigma$;
 - 7) $V(\Sigma)$ and $V(p, \Sigma)$ (when $0 \notin \Sigma$) the $\Sigma \times J$ -matrixes defined, respectively, by $V(\Sigma)[s] := V[s]$ and $V(p, \Sigma) := V(p)[s]$, for each $s \in \Sigma$, where $V(p) := ((p[s] \cdot e)v_j[s])$;
 - 8) $W(\Sigma, q)$ and $W(\Sigma, p, q)$ (when $0 \notin \Sigma$) the $\Sigma' \times J$ -matrixes defined, respectively, by $W(\Sigma, q)[0] := W(\Sigma, p, q)[0] := -q$, and by $W(\Sigma, q)[s] := V[s]$ and $W(\Sigma, p, q)[s] := V(p)[s]$ for every $s \in \Sigma$. We let $W(q) := W(S, q)$ and $W(p, q) := W(S, p, q)$;
 - 9) $(\mathbb{R}_+^L)^\Sigma := \{x \in (\mathbb{R}^L)^\Sigma : x \geq 0\}$, $\mathbb{R}_+^\Sigma := \{x \in \mathbb{R}^\Sigma : x \geq 0\}$,
 $(\mathbb{R}_{++}^L)^\Sigma := \{x \in (\mathbb{R}^L)^\Sigma : x >> 0\}$, $\mathbb{R}_{++}^\Sigma := \{x \in \mathbb{R}^\Sigma : x >> 0\}$.

Given prices $(p, q) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J$, each agent $i \in I$ detaining the information Σ (for $\Sigma \subset S$) has the following budget set, whose elements are called strategies:

$$B_i(\Sigma, p, q) := \{(x, z) \in X(\Sigma) \times \mathbb{R}^J : (p_\square[x - e_i(\Sigma)])(\Sigma') \leq W(\Sigma, p, q)z\}.$$

Given the structures (S_i) , $(\Sigma_i) \leq (S_i)$ and prices $(p, q) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J$, an allocation $(x_i) \in \Pi_{i=1}^m X(\Sigma_i)$ is attainable if $\sum_{i=1}^m (x_i - e_i(\Sigma_i))[\underline{\mathbf{S}}'] = 0$. We let:

$$\begin{aligned} \mathcal{A}(\Sigma_i) &:= \{x := (x_i) \in \Pi_{i=1}^m X(\Sigma_i) : \sum_{i=1}^m (x_i - e_i(\Sigma_i))[\underline{\mathbf{S}}'] = 0\}, \\ \mathcal{Z} &:= \{(z_i) \in (\mathbb{R}^J)^m : \sum_{i=1}^m z_i = 0\} \text{ and} \\ \mathcal{A}((\Sigma_i), p, q) &:= \{[(x_i, z_i)] \in \Pi_{i=1}^m B_i(\Sigma_i, p, q) : (x_i) \in \mathcal{A}(\Sigma_i), (z_i) \in \mathcal{Z}\} \end{aligned}$$

be, respectively, the sets of attainable allocations, portfolios and strategies.

The economy described above for a given payoff matrix V and a given structure (S_i) of information signals $S_i \subset S$, which each agent $i \in I$ receives privately at $t = 0$, is denoted by $\mathcal{E}[V, (S_i)]$ and yields the following concept of equilibrium.

Definition 2 *Given a structure $[V, (S_i)]$, the economy $\mathcal{E}[V, (S_i)]$ is called standard if it meets Assumptions A1 to A4; a price system $(p^*, q^*) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J$, a refinement $(\Sigma_i) \leq (S_i)$ and a collection of strategies $(x_i^*, z_i^*) \in B_i(\Sigma_i, p^*, q^*)$, for $i = 1, \dots, m$, define a (competitive) equilibrium of the economy $\mathcal{E}[V, (S_i)]$ if:*

- (a) $\forall i \in I, B_i(\Sigma_i, p^*, q^*) \cap P_i^{\Sigma_i}(x_i^*) \times \mathbb{R}^J = \emptyset$;
- (b) $\sum_{i \in I} (x_i^* - e_i(\Sigma_i))[\underline{\mathbf{S}}'] = 0$;
- (c) $\sum_{i \in I} z_i^* = 0$.

We now recall some basic properties of the model.

2.3 Basic arbitrage properties

First, we extend the classical notions of arbitrage and no-arbitrage price to asymmetric information economies along the following Definition.

Definition 3 *Given a structure $[V, (\Sigma_i)]$, a price $q \in \mathbb{R}^J$ is said to be a common no-arbitrage price of the structure $[V, (\Sigma_i)]$, or the structure $[V, (\Sigma_i)]$ to be q -arbitrage-free, if one of the following equivalent assertions holds:*

- (a) $\nexists (i, z) \in I \times \mathbb{R}^J : W(\Sigma_i, q)z > 0$;
- (b) $\nexists (i, z, p) \in I \times \mathbb{R}^J \times (\mathbb{R}^L)^{S'} : W(\Sigma_i, p, q)z > 0$ and $p[s] \cdot e > 0, \forall s \in S$;
- (c) $\forall i \in I, \exists \lambda_i \in \mathbb{R}_{++}^{\Sigma_i}$ (called individual state price), such that $q = {}^t \lambda_i V(\Sigma_i)$. We denote by $Q_c[V, (\Sigma_i)]$ the set of common no-arbitrage prices of $[V, (\Sigma_i)]$. The structure $[V, (\Sigma_i)]$ is said to be arbitrage-free if $Q_c[V, (\Sigma_i)] \neq \emptyset$.

Given a structure $[V, (S_i)]$ and $q \in \mathbb{R}^J$, we say that q is a no-arbitrage price (of $\mathcal{E}[V, (S_i)]$) if there exists a refinement (Σ_i) of (S_i) , such that $q \in Q_c[V, (\Sigma_i)]$. We denote by $Q[V, (S_i)]$ the set of no-arbitrage prices and by $\mathcal{S}[V, (S_i), q]$ (resp. $\mathcal{S}[V, (S_i)]$) the set of q -arbitrage-free (resp. arbitrage-free) refinements of (S_i) . When no confusion is possible, we may omit the reference to V in all notations.

Remark 1 The equivalence between Assertions (a) and (c) above is standard (Magill and Quinzii, 1996 [9]). That with Assertion (b) is immediate. Indeed, if Assertion (c) holds, let $(i, \lambda_i, p) \in I \times \mathbb{R}_{++}^{\Sigma_i} \times (\mathbb{R}^L)^{S'}$ be given, s.t. $q = {}^t\lambda_i V(\Sigma_i)$ and $p[s] \cdot e > 0$ for each $s \in S$. Then, $\mu_i \in \mathbb{R}_{++}^{\Sigma_i}$, defined by $\mu_i[s] := \frac{\lambda_i[s]}{p[s] \cdot e}$ for each $s \in \Sigma_i$, satisfies $q = {}^t\mu_i V(\Sigma_i, p)$ and, by standard separation, there is no $(i, z) \in I \times \mathbb{R}^J$ s.t. $W(\Sigma_i, p, q)z > 0$; hence, Assertion (b) holds. If Assertion (b) holds, let $\bar{p} \in (\mathbb{R}^L)^{S'}$ satisfy $\bar{p}[s] \cdot e = 1$, for each $s \in S$, then, there is no $(i, z) \in I \times \mathbb{R}^J$, s.t. $W(\Sigma_i, \bar{p}, q) = W(\Sigma_i, q)z > 0$, i.e., Assertion (a) holds. \square

Remark 2 We recall from [3] that a symmetric structure is arbitrage-free, hence, $\mathcal{S}[V, (S_i)]$ and $Q[V, (S_i)]$ are nonempty. Moreover, the above Definitions extend to asymmetric information the classical notions of price and equilibrium.

This paper shows arbitrage-free structures characterize competitive equilibria in any economy $\mathcal{E}[V, (S_i)]$. Departing from the generic existence of a fully-revealing rational expectations equilibrium along Radner, 1979 [11], this result allows for the broadest pattern of asymmetry at equilibrium and extends a classical property of symmetric information. Indeed, Geanakoplos-Polemarchakis, 1986 [8] show, in a symmetric information model, that equilibrium with numeraire assets exists, whenever markets preclude arbitrage, and is generically locally unique. Before proving each arbitrage-free refinement can be embedded into equilibrium, the Claims below recall a converse and other basic results.

Claim 1 *Let the structure $[V, (S_i)]$, prices $(p, q) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J$, a refinement $(\Sigma_i) \leq (S_i)$ and strategies $[(x_i, z_i)] \in \Pi_{i=1}^m B_i(\Sigma_i, p, q)$ be given. Under Assumption A1, if the collection $((p, q), (\Sigma_i), [(x_i, z_i)])$ meets Condition (a) of the Definition of equilibrium, then, $q \in Q_c[V, (\Sigma_i)]$, hence, $(\Sigma_i) \in \mathcal{S}[V, (S_i)]$.*

Proof Given a structure $[V, (S_i)]$, assume that $(p, q) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J$, $(\Sigma_i) \leq (S_i)$ and $[(x_i, z_i)] \in \Pi_{i=1}^m B_i(\Sigma_i, p, q)$ satisfy Condition (a) of Definition 2, with $q \notin Q_c[V, (\Sigma_i)]$, that is, $W(\Sigma_i, q)z > 0$, for some $(i, z) \in I \times \mathbb{R}^J$. The reader will readily check from Assumption A1 that $p[s] \cdot e > 0$, for every $s \in S'$. Then, $W(\Sigma_i, p, q)z > 0$, and we let $\bar{s} \in \Sigma'_i$, be such that $W(p, q)[\bar{s}] \cdot z > 0$ and $\lambda := p[\bar{s}] \cdot e / W(p, q)[\bar{s}] \cdot z$. Referring to Assumption A1, $(x_i + e_{\bar{s}}, z_i + \lambda z) \in B_i(\Sigma_i, p, q) \cap P_i^{\Sigma_i}(x_i) \times \mathbb{R}^J$, which contradicts the initial assumptions. \square

Claim 2 *Given $q \in \mathbb{R}^J$ and a structure $[V, (S_i)]$, the Assertions below hold:*
(i) *A refinement $(\Sigma_i) \leq (S_i)$ is arbitrage-free if and only if it satisfies the following “AFAO” Condition (“Absence of Future Arbitrage Opportunities”): $\nexists (z_i) \in \mathcal{Z}$, $\nexists j \in I$, s.t. $V(\Sigma_j)z_j > 0$ and $V(\Sigma_i)z_i \geq 0, \forall i \in I$;*
(ii) *$\mathcal{S}[V, (S_i), q] \neq \emptyset$ if and only if $q \in Q[V, (S_i)]$. If so, $\mathcal{S}[V, (S_i), q]$ contains a coarsest element, denoted by $(\mathbf{S}_i(q))$, which is said to be “revealed” by price q ;*
(iii) *$\mathcal{S}[V, (S_i)]$ contains a coarsest element, denoted by $(\bar{\mathbf{S}}_i)$, which is revealed by any $\bar{q} \in Q_c[V, (\bar{\mathbf{S}}_i)]$ and equal to (S_i) if and only if (S_i) is arbitrage-free.*

Proof See Cornet-De Boisdeffre, 2002 [3]. \square

3 The existence theorem

We now state the theorem, bound the economy, and infer the existence of equilibrium from a standard fixed-point-like argument in a compact economy.

Theorem 1 *Let a standard economy $\mathcal{E}[V, (S_i)]$ and $(\Sigma_i) \in \mathcal{S}[V, (S_i)]$ be given. Then, there exist prices $(p, q) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J$ and strategies $[(x_i, z_i)] \in \prod_{i=1}^m B_i(\Sigma_i, p, q)$, such that $((p, q), (\Sigma_i), [(x_i, z_i)])$ is an equilibrium of the economy $\mathcal{E}[V, (S_i)]$ and $p[s] \cdot e > 0$, for each $s \in S'$.*

Henceforth, a structure $[V, (S_i)]$, a corresponding standard economy $\mathcal{E}[V, (S_i)]$ and a refinement $(\Sigma_i) \in \mathcal{S}[V, (S_i)]$ are given. We let $e_i := e_i(\Sigma_i)$, for each $i \in I$, and assume, at no cost, that $S = \cup_{i=1}^m \Sigma_i$. Denoting $X := \prod_{i=1}^m X(\Sigma_i)$, we define, for each $i \in I$, a correspondence $P_i : X \rightarrow X(\Sigma_i)$ by $P_i(x) := P_i^{\Sigma_i}(x_i)$, for every $x := (x_i) \in X$, which is open, from Assumption A3.

3.1 Bounding strategies

The two Lemmas below serve to bound strategies in the equilibrium problem. Under asymmetric information, the issue is complex. First, spot markets will not clear, in general, in states $s \notin \underline{S}'$, where one agent, at least, has no budget constraint: to bound consumptions in such states, a positivity constraint is introduced on commodity prices. Second, portfolios will no longer bound in \mathbb{R}^J , but in specifically tailored sub-vector spaces. Thus, for each $i \in I$, we let:

$$\begin{aligned} Z_i^o &:= \{z \in \mathbb{R}^J : V[s] \cdot z = 0, \forall s \in \Sigma_i\} \text{ and denote by } Z_i^{o\perp} \text{ its orthogonal;} \\ Z^o &:= \sum_{i=1}^m Z_i^o \text{ and denote by } Z^{o\perp} = \cap_{i \in I} Z_i^{o\perp} \text{ its orthogonal.} \end{aligned}$$

With symmetric information, $Z_i^o = Z^o$, for each $i \in I$, and $Z^o := \{0\}$ after redundant assets are eliminated: attainable strategies may be bounded with transfers by simple market-clearance arguments. With asymmetric information, this is no longer the case. A specific elimination rule for each agent and a fitted clearance condition for assets are needed. Namely, we have to look for portfolios $(z_i) \in \prod_{i=1}^m Z_i^{o\perp}$ such that $(\sum_{i=1}^m z_i) \in Z^o$, and for asset prices in the sub-space $Z^{o\perp}$, which includes the set $Q[V, (S_i)]$ of no-arbitrage prices. The Appendix discloses why such specific rules are needed to bound strategies under asymmetric information, which simplify to a standard problem in the symmetric case. Thus, we let $\varepsilon \in]0, \frac{1}{L}[$ be given and consider the following price sets:

$$\begin{aligned} \Delta &:= \{p \in (\mathbb{R}^L)^{S'} : \|p[s]\| \leq 1, \forall s \in S', p^l[\bar{s}] \geq \varepsilon, \forall (l, \bar{s}) \in \{1, \dots, L\} \times S \setminus \underline{S}\}; \\ \Delta^* &:= \{p \in \Delta : \|p[s]\| = 1, \forall s \in \underline{S} := \cap_i S_i\}; \\ \Delta_\delta &:= \{p \in \Delta : p[s] \cdot e \geq \delta, \forall s \in S\} \neq \emptyset, \text{ for each } \delta \in]0, \varepsilon[; \\ Q &:= \{q \in Z^{o\perp} : \|q\| \leq 1\}; \\ \Pi &:= \Delta \times Q; \Pi^* := \Delta^* \times Q \text{ and } \Pi_\delta := \Delta_\delta \times Q. \end{aligned}$$

We denote by $\mathbf{1}$ the vector of $\mathbb{R}^{S'}$ whose components are all equal to one and consider, for every $i \in I$, $(p, q) \in \Pi$, $n \geq 1$, the following strategy sets:

$$\begin{aligned}\overline{B}_i(p, q) &:= \{(x, z) \in X(\Sigma_i) \times Z_i^{o\perp} : (p_\square[x - e_i])[\Sigma'_i] \leq W(\Sigma_i, p, q)z + \mathbf{1}[\Sigma'_i]\}; \\ \overline{B}_i^n(p, q) &:= \{(x, z) \in \overline{B}_i(p, q) : \|z\| \leq n\}; \\ \overline{\mathcal{A}}(p, q) &:= \{[(x_i, z_i)] \in \Pi_{i=1}^m \overline{B}_i(p, q) : (x_i) \in \mathcal{A}(\Sigma_i), \sum_{i=1}^m z_i \in Z^o\}; \\ \text{and } \overline{\mathcal{A}}^n(p, q) &:= \{[(x_i, z_i)] \in \Pi_{i=1}^m \overline{B}_i^n(p, q) : (x_i) \in \mathcal{A}(\Sigma_i), \sum_{i=1}^m z_i \in Z^o\};\end{aligned}$$

for any price system $(p, q) \in \Pi$ and collection $[(x_i, z_i)] \in \Pi_{i=1}^m \overline{B}_i(p, q)$, we henceforth denote $x := (x_i)$ and $\overline{x} := \sum_{i=1}^m x_i$, $z := (z_i)$ and $\overline{z} := \sum_{i=1}^m z_i$, and $(x, z) := [(x_i, z_i)] \in \Pi_{i=1}^m \overline{B}_i(p, q)$.

We can now state Lemma 1.

Lemma 1 *Given the structure $(\Sigma_i) \in \mathcal{S}[V, (S_i)]$ and the above definitions of the sets Π , Π_δ , $\overline{\mathcal{A}}(p, q)$, $\overline{\mathcal{A}}^n(p, q)$ (for $((p, q), \delta) \in \Pi \times]0, \varepsilon[$ and $n \geq 1$), the following Assertions hold:*

- (i) $\forall n \geq 1, \exists r_n > 0 : [(p, q) \in \Pi \ \& \ (x, z) \in \overline{\mathcal{A}}^n(p, q)] \implies [\|x\| + \|z\| < r_n];$
- (ii) $\forall \delta \in]0, \varepsilon[, \exists r_\delta > 0 : [(p, q) \in \Pi_\delta \ \& \ (x, z) \in \overline{\mathcal{A}}(p, q)] \implies [\|x\| + \|z\| < r_\delta].$

Proof: See the Appendix. \square

For each $s \in \underline{\mathbf{S}} := \cap_i S_i$, we let: $\mathcal{P}_s := \{p_s \in \mathbb{R}^L : \exists p \in \Delta^* \text{ s.t. } p_s = p[s], \exists i \in I, \exists x := (x_i) \in \mathcal{A}(\Sigma_i) \text{ s.t. } (y_i \in P_i(x) \text{ and } y_i[S' \setminus \{s\}] = x_i[S' \setminus \{s\}]) \text{ imply } (p[s] \cdot y_i[s] \geq p[s] \cdot x_i[s] \geq p[s] \cdot e_i[s])\};$
 $\mathcal{P} := \{p \in \Delta^* : p[s] \in \mathcal{P}_s, \forall s \in \underline{\mathbf{S}}\}.$

Lemma 2 *The following Assertions hold:*

- (i) *for each $s \in \underline{\mathbf{S}}$, \mathcal{P}_s is closed, hence, \mathcal{P}_s and \mathcal{P} are compact sets;*
- (ii) *there exists $\underline{\delta} \in]0, \varepsilon[$, such that $\mathcal{P} \subset \Delta_{\underline{\delta}}$.*

Proof See the Appendix. \square

The above Lemmas will permit to bound the economy. We henceforth set as given $\underline{\delta} \in]0, \varepsilon[$, satisfying Condition (ii) of Lemma 2, and $\underline{r} := r_{\underline{\delta}}$, satisfying Condition (ii) of Lemma 1 for that bound $\underline{\delta}$. We also set as given an integer $n > \underline{r}$ and some bound $r := r_n > \underline{r} + n$, meeting Condition (i) of Lemma 1 relative to n , and let, for every $(i, (p, q)) \in I \times \Pi$:

$$\begin{aligned}X_i &:= \{x \in X(\Sigma_i) : \|x\| \leq r\}; \\ Z_i &:= \{z \in Z_i^{o\perp} : \|z\| \leq n\}; \\ B_i(p, q) &:= B_i(\Sigma_i, p, q) \cap (X_i \times Z_i) \subset \overline{B}_i^n(p, q); \\ \mathcal{A} &:= \mathcal{A}(\Sigma_i) \cap \Pi_{i=1}^m X_i; \\ \mathcal{A}(p, q) &:= \{(x, z) \in \Pi_{i=1}^m B_i(p, q) : x \in \mathcal{A}, \overline{z} \in Z^o\} \subset \overline{\mathcal{A}}^n(p, q) \text{ for the above } n\end{aligned}$$

We can now prove Theorem 1.

3.2 The existence proof

We now prove the existence of equilibrium from a fixed-point-like argument. Along Bergstrom, 1976 [1] and Florenzano, 1999 [5], we define, for each $i \in I$ and every $(p, q) \in \Pi$, specifically tailored correspondences on the compact sets $(X_i, Z_i, \Pi \dots)$, and the following “augmented” budget sets:

$$\begin{aligned} B'_i(p, q) &:= \{(x, z) \in X_i \times Z_i : (p_\square[x - e_i])[\Sigma'_i] \leq W(\Sigma_i, p, q)z + \gamma_{(p, q)}[\Sigma'_i]\}; \\ B''_i(p, q) &:= \{(x, z) \in X_i \times Z_i : (p_\square[x - e_i])[\Sigma'_i] < W(\Sigma_i, p, q)z + \gamma_{(p, q)}[\Sigma'_i]\}, \end{aligned}$$

where $\gamma_{(p, q)} \in \mathfrak{R}_+^{S'}$ is defined by: $\gamma_{(p, q)}[0] := 1 - \min(1, \|p[0]\| + \|q\|)$, $\gamma_{(p, q)}[s] := 1 - \|p[s]\|$ for every $s \in \underline{S} := \cap_i S_i$ and $\gamma_{(p, q)}[S \setminus \underline{S}] := 0$.

Claim 3 *For every $i \in I$, and every $(p, q) \in \Pi$, $B''_i(p, q) \neq \emptyset$.*

Proof Let $i \in I$ and $(p, q) \in \Pi$ be given. From Assumption A2, there exists $x \in X_i$, such that $(p_\square[x - e_i])[\Sigma'_i] \leq 0$, with a strict inequality in any state $s \in \Sigma_i$ such that $p[s] \neq 0$ (in particular, for each $s \in \Sigma_i \setminus \underline{S}$). If $p[0] \neq 0$, or if $p[0] = 0$ and $q = 0$, then, obviously, $(x, 0) \in B''_i(p, q)$. Alternatively, if $p[0] = 0$ and $q \neq 0$, recalling that $q \in Z^{o\perp} = \cap_{i \in I} Z_i^{o\perp}$ yields $(x, z) \in B''_i(p, q)$, for some small $z \in Z_i := \{z \in Z_i^{o\perp} : \|z\| \leq n\}$, such that $q \cdot z < 0$. \square

Claim 4 *For every $i \in I$, B''_i is convex-valued and lower semicontinuous.*

Proof Let $(i, (p, q)) \in I \times \Pi$ be given. The convexity of $B''_i(p, q)$ is obvious and implies, from Claim 3, $B'_i(p, q) = \overline{B''_i(p, q)}$. Then, B''_i is lower semicontinuous, as is standard, for having an open graph. \square

Claim 5 *For every $i \in I$, B'_i is convex-valued and upper semicontinuous.*

Proof For each $i \in I$, B'_i is obviously non-empty convex-valued, and upper semicontinuous, as is standard, for having a closed graph in a compact set. \square

We now introduce an additional agent $i = 0$, representing the market, and a reaction correspondence for each agent, defined on the convex compact set $\Theta := \Pi \times (\prod_{i=1}^m X_i \times Z_i)$. We let, for each $i \in I$ and every $((p, q), (x, z)) \in \Theta$:

$$\Psi_i((p, q), (x, z)) := \begin{cases} B'_i(p, q) & \text{if } (x_i, z_i) \notin B'_i(p, q) \\ B'_i(p, q) \cap P_i(x) \times Z_i & \text{if } (x_i, z_i) \in B'_i(p, q) \end{cases}.$$

The price correspondence Ψ_0 satisfies, for every $((p, q), (x, z)) \in \Theta$:

$$\begin{aligned} \Psi_0((p, q), (x, z)) &:= \\ \{(p', q') \in \Pi : (p' - p) \cdot \sum_{i=1}^m (x_i - e_i) + (q' - q) \cdot \sum_{i=1}^m z_i > 0\}. \end{aligned}$$

Claim 6 *For each $i \in \{0, 1, \dots, m\}$, Ψ_i is lower semicontinuous.*

Proof First, the correspondence Ψ_0 is lower semicontinuous for having an open graph. Second, we let $i \in I$ and $\omega := ((p, q), (x, z)) \in \Theta$ be given and consider separately the two alternatives $(x_i, z_i) \notin B'_i(p, q)$ and $(x_i, z_i) \in B'_i(p, q)$ and show that, in both cases, Ψ_i is lower semicontinuous at ω .

- Assume, first, that $(x_i, z_i) \notin B'_i(p, q)$. Then, $\Psi_i(\omega) = B'_i(p, q)$.

Let V be an open set in $X_i \times Z_i$, such that $V \cap B'_i(p, q) \neq \emptyset$. It follows from the convexity of $B'_i(p, q)$ and the non-emptiness of the open set $B''_i(p, q)$ that $V \cap B''_i(p, q) \neq \emptyset$. From Claim 4, there exists a neighborhood U of (p, q) in Π , such that $V \cap B'_i(p', q') \supset V \cap B''_i(p', q') \neq \emptyset$, for every $(p', q') \in U$.

Since $B'_i(p, q)$ is nonempty, closed, convex in the compact set $X_i \times Z_i$, there exist two open sets V_1 and V_2 in $X_i \times Z_i$, such that $(x_i, z_i) \in V_1$, $B'_i(p, q) \subset V_2$ and $V_1 \cap V_2 = \emptyset$.⁴ From Claim 5, there exists a neighborhood $U_1 \subset U$ of (p, q) , such that $B'_i(p', q') \subset V_2$, for every $(p', q') \in U_1$. Let $W = U_1 \times \prod_{j=1}^m W_j$, where $W_i := V_1$ and $W_j := X_j \times Z_j$, for every $j \in I \setminus \{i\}$. Then, W is a neighborhood of ω , such that $\Psi_i(\omega') = B'_i(p', q')$, and, from above, $V \cap \Psi_i(\omega') \neq \emptyset$, for every $\omega' := ((p', q'), (x', z')) \in W$. This proves the lower semicontinuity of Ψ_i at ω .

- Assume, now, that $(x_i, z_i) \in B'_i(p, q)$, i.e., $\Psi_i(\omega) = B'_i(p, q) \cap P_i(x) \times Z_i$.

The lower semicontinuity of Ψ_i at ω is immediate if $\Psi_i(\omega) = \emptyset$. Assume $\Psi_i(\omega) \neq \emptyset$. We recall that P_i (from Assumptions A3) and B''_i are open. The correspondence $((p', q'), (x', z')) \in \Theta \mapsto B''_i(p', q') \cap P_i(x') \times Z_i \subset B'_i(p', q')$ is lower semicontinuous with open values, as a standard corollary. It follows from the latter inclusions in the compact sets $B'_i(p', q')$ that Ψ_i is lower semicontinuous at ω . \square

Claim 7 *The correspondences Ψ_i (for $i \in \{0, 1, \dots, m\}$) admit an element $((p^*, q^*), (x^*, z^*))$ of Θ , such that:*

- (i) $\forall (p, q) \in \Pi$, $(p^* - p) \cdot \sum_{i=1}^m (x_i^* - e_i) + (q^* - q) \cdot \sum_{i=1}^m z_i^* \geq 0$;
- (ii) $\forall i \in I$, $(x_i^*, z_i^*) \in B'_i(p^*, q^*)$ and $B''_i(p^*, q^*) \cap P_i(x^*) \times Z_i = \emptyset$.

Proof We recall the following fixed-point-like theorem, due to Gale and Mas-Colell, 1975-1979 [9, 10]: “Given $X = \prod_{i=1}^m X_i$, where X_i is a non-empty compact convex subset of \mathbb{R}^n , let $\varphi_i : X \rightarrow X_i$ be m convex (possibly empty) valued correspondences, which are lower semicontinuous. Then there exists x in X such that for each i either $x_i \in \varphi_i(x)$ or $\varphi_i(x) = \emptyset$ ”.

From above, the correspondences $\Psi_0 : \Theta \rightarrow \Pi$ and $\Psi_i : \Theta \rightarrow X_i \times Z_i$ (for $i = 1, \dots, m$) satisfy the conditions of the theorem and, hence, admit an element $((p^*, q^*), (x^*, z^*)) \in \Theta$, such that either $(p^*, q^*) \in \Psi_0((p^*, q^*), (x^*, z^*))$ or $\Psi_0((p^*, q^*), (x^*, z^*)) = \emptyset$, and, for each $i \in I$, either $(x_i^*, z_i^*) \in \Psi_i((p^*, q^*), (x^*, z^*))$

⁴e.g., let V_1 be the open ball centered on (x_i, z_i) of radius $\frac{\alpha}{2}$, where $\alpha > 0$ is the distance from (x_i, z_i) to $B'_i(p, q)$, and V_2 be a finite union of open balls of radius $\frac{\alpha}{2}$, centered on elements of $B'_i(p, q)$, containing $B'_i(p, q)$.

or $\Psi_i((p^*, q^*), (x^*, z^*)) = \emptyset$. By construction, $(p^*, q^*) \notin \Psi_0((p^*, q^*), (x^*, z^*))$ and, for each $i \in I$, $(x_i^*, z_i^*) \notin \Psi_i((p^*, q^*), (x^*, z^*))$, since $x_i^* \notin P_i(x^*)$. Hence, $\Psi_0((p^*, q^*), (x^*, z^*)) = \emptyset$, which yields the above Assertion (i) and, for every $i \in I$, $\Psi_i((p^*, q^*), (x^*, z^*)) = \emptyset$, which yields the above Assertion (ii). \square

Claim 8 $\bar{z}^* := (\sum_{i=1}^m z_i^*) \in Z^o$.

Proof Assume, by contraposition, $\bar{z}^* \notin Z^o$. Then, from Claim 7-(i), $q \cdot \bar{z}^* \leq q^* \cdot \bar{z}^*$, for every $q \in Q := \{q \in Z^{o\perp} : \|q\| \leq 1\} \supset \{0\}$, which implies $q^* \cdot \bar{z}^* > 0$ and $\|q^*\| = 1$, hence, $\gamma_{(p^*, q^*)}[0] = 0$.

From Claim 7-(ii), for each $i \in I$, $(x_i^*, z_i^*) \in B'_i(p^*, q^*)$, whose budget constraint in state $s = 0$ is $p^*[0] \cdot (x_i^*[0] - e_i[0]) \leq -q^* \cdot z_i^*$. Then, summing up for $i = 1, \dots, m$ yields, from above, $p^*[0] \cdot \sum_{i=1}^m (x_i^*[0] - e_i[0]) \leq -q^* \cdot \sum_{i=1}^m z_i^* < 0$, contradicting Claim 7-(i) (written for $(p, q) \in \Pi$, such that $p[0] = 0$, $p[S] = p^*[S]$, $q = q^*$), which implies $p^*[0] \cdot \sum_{i=1}^m (x_i^*[0] - e_i[0]) \geq 0$. \square

Remark 4 Since $Q \subset Z^{o\perp}$, from Claim 8, Claim 7-(i) may now be written $(p^* - p) \cdot \sum_{i=1}^m (x_i^* - e_i) \geq 0$, for all $p \in \Delta$, and there exists $z' := (z'_i) \in \Pi_{i=1}^m Z_i^o$ such that $\sum_{i=1}^m z_i^* = \sum_{i=1}^m z'_i$. For each $i \in I$, we henceforth let $z_i := z_i^* - z'_i$, which satisfies $W(\underline{\mathbf{S}}, p^*, q^*)z_i^* = W(\underline{\mathbf{S}}, p^*, q^*)z_i$ (since $q^* \in Z_i^{o\perp}$ and $z'_i \in Z_i^o$) and $\sum_{i=1}^m z_i = 0$.

Claim 9 $x^* = (x_i^*) \in \mathcal{A}(\Sigma_i)$, i.e., $\sum_{i=1}^m (x_i^*[s] - e_i[s]) = 0, \forall s \in \underline{\mathbf{S}}'$.

Proof Assume, by contraposition, that $\sum_{i=1}^m (x_i^*[s] - e_i[s]) \neq 0$, for some $s \in \underline{\mathbf{S}}'$. Applying Claim 7-(i) to prices $p \in \Delta$ such that $p[S' \setminus \{s\}] = p^*[S' \setminus \{s\}]$ yields: $p^*[s] = \sum_{i=1}^m (x_i^*[s] - e_i[s]) / \|\sum_{i=1}^m (x_i^*[s] - e_i[s])\|$, hence, $\gamma_{(p^*, q^*)}[s] = 0$ and $p^*[s] \cdot \sum_{i=1}^m (x_i^*[s] - e_i[s]) > 0$. Then, Claim 7-(ii), yields, for each $i \in I$, $p^*[s] \cdot (x_i^*[s] - e_i[s]) \leq W(p^*, q^*)[s] \cdot z_i^*$, and, summing up for $i = 1, \dots, m$ yields, from Remark 4:

$p^*[s] \cdot \sum_{i=1}^m (x_i^*[s] - e_i[s]) \leq \sum_{i=1}^m W(p^*, q^*)[s] \cdot z_i^* = W(p^*, q^*)[s] \cdot \sum_{i=1}^m z_i = 0$, hence, the desired contradiction (with $p^*[s] \cdot \sum_{i=1}^m (x_i^*[s] - e_i[s]) > 0$). \square

Claim 10 $(x^*, z^*) := [(x_i^*, z_i^*)] \in \overline{\mathcal{A}}^n(p^*, q^*)$, hence, $\|x^*\| + \|z^*\| < r$.

Proof Recalling the definitions on pages 7-8, and of $r, n, \overline{\mathcal{A}}^n(p^*, q^*), X_i, Z_i$ (for $i \in I$), Claim 7-(ii) yields, for every $i \in I$, $(x_i^*, z_i^*) \in B'_i(p^*, q^*) \subset \overline{B}_i^n(p^*, q^*)$, and, from Claims 8-9, $(x^*, z^*) \in \overline{\mathcal{A}}^n(p^*, q^*)$. Then, from Lemma 1, $\|x^*\| + \|z^*\| < r$. \square

Claim 11 For each $i \in I$, (x_i^*, z_i^*) is optimal in $B'_i(p^*, q^*)$.

Proof Let $i \in I$ be given. From Claim 7-(ii), $(x_i^*, z_i^*) \in B'_i(p^*, q^*)$ and $B''_i(p^*, q^*) \cap P_i(x^*) \times Z_i = \emptyset$.

By contraposition, assume there exists $(x_i, z_i) \in B'_i(p^*, q^*) \cap P_i(x^*) \times Z_i$. From Claim 10, the strict inequality $\|x_i^*\| + \|z_i^*\| < r$ holds and, from Assumption A4 and the convexity all sets, the better strategy (x_i, z_i) may be chosen “sufficiently close” to (x_i^*, z_i^*) so that $\|x_i\| + \|z_i\| < r$.

From Claim 3, there exists $(x'_i, z'_i) \in B''_i(p^*, q^*) \subset B'_i(p^*, q^*)$. Since $B'_i(p^*, q^*)$ is convex, $(x_i^p, z_i^p) := [\frac{1}{p}(x'_i, z'_i) + (1 - \frac{1}{p})(x_i, z_i)] \in B'_i(p^*, q^*)$, whereas, by construction, $(x_i^p, z_i^p) \in B''_i(p^*, q^*)$, for every $p > 0$. From Assumption A3, P_i is open valued and there exists $P > 0$, such that $(x_i^P, z_i^P) \in P_i(x^*) \times Z_i$. Hence, $(x_i^P, z_i^P) \in B''_i(p^*, q^*) \cap P_i(x^*) \times Z_i$, which contradicts Claim 7-(ii). \square

Claim 12 $\gamma_{(p^*, q^*)} = 0$, that is, $B'_i(p^*, q^*) = B_i(p^*, q^*)$, for every $i \in I$.

Proof Let $(i, s) \in I \times \underline{\mathbf{S}}'$ be given.

We show first that: $p^*[s] \cdot (x_i^*[s] - e_i[s]) = W(p^*, q^*)[s] \cdot z_i^* + \gamma_{(p^*, q^*)}[s]$.

Indeed, from Claim 7-(ii), $p^*[s] \cdot (x_i^*[s] - e_i[s]) \leq W(p^*, q^*)[s] \cdot z_i^* + \gamma_{(p^*, q^*)}[s]$, whereas, from Claim 10, $\|x_i^*\| < r$ and, from Assumptions A1 & A4, there exists $x_i \in P_i(x^*)$, such that $x_i[s'] = x_i^*[s']$ for every $s' \neq s$. From Assumption A4, if we had $p^*[s] \cdot (x_i^*[s] - e_i[s]) < W(p^*, q^*)[s] \cdot z_i^* + \gamma_{(p^*, q^*)}[s]$, then, x_i could be chosen “sufficiently close” to x_i^* so that $p^*[s] \cdot (x_i[s] - e_i[s]) \leq W(p^*, q^*)[s] \cdot z_i^* + \gamma_{(p^*, q^*)}[s]$ and this would contradict Claim 11.

Summing up on $i \in I$ the above relations, for each $(i, s) \in I \times \underline{\mathbf{S}}'$, yields, from Claims 8 and 9, and from Remark 4, $\gamma_{(p^*, q^*)} = 0$, i.e., $B'_i(p^*, q^*) = B_i(p^*, q^*)$, for every $i \in I$. \square

Claim 13 $\forall s \in S \setminus \underline{\mathbf{S}}, \|p^*[s]\| \geq \varepsilon, \forall s \in \underline{\mathbf{S}}, \|p^*[s]\| = 1, \forall s \in S', p^*[s] \cdot e > 0$.

Proof Let $s \in S'$ be given. If $s \in \underline{\mathbf{S}}$, Claim 12 yields $\|p^*[s]\| = 1$. If $s \in S \setminus \underline{\mathbf{S}}$, $\|p^*[s]\| \geq \varepsilon$ and $p^*[s] \cdot e \geq \varepsilon > 0$, from the definition of Δ . If $s \in \underline{\mathbf{S}}'$, Claim 10 and Assumptions A1-A4 yield $x_1 \in P_1(x^*)$ and $\lambda > 0$, such that $\|x_1\| < r$, $x_1[S' \setminus \{s\}] = x_1^*[S' \setminus \{s\}]$ and $x_1[s] = x_1^*[s] + \lambda e$, which implies $p^*[s] \cdot e > 0$, from Claim 7-(ii) and Claim 11. \square

Claim 14 $p^* \in \Delta_{\underline{\delta}}$ and $((p^*, q^*), (\Sigma_i), (x^*, z))$, where $z := (z_i)$ is defined as in Remark 4, is an equilibrium of $\mathcal{E}[V, (S_i)]$. Hence, Theorem 1 holds.

Proof We show, first, that $p^* \in \mathcal{P}$, i.e., $p^*[s] \in \mathcal{P}_s$, for every $s \in \underline{\mathbf{S}}$. Indeed, from Claim 13, $p^* \in \Delta^* := \{p \in \Delta : \|p^*[s]\| = 1, \forall s \in \underline{\mathbf{S}}\}$ and, from Claim 9, $x^* \in \mathcal{A}(\Sigma_i)$. Let $s \in \underline{\mathbf{S}}$ be given. Referring the reader to Remark 4 and to the proof of Claim 12, one has $V(p^*)[s] \cdot z_i = p^*[s] \cdot (x_i^* - e_i)[s]$, for each $i \in I$, with $\sum_{i=1}^m z_i = 0$. Thus, there exists $i \in I$, such that $V(p^*)[s] \cdot z_i \geq 0$, and we let the reader check from Claims 10, 11 and 12, and from Assumption A4, that the triple $(p[s], x^*, i)$ meets the conditions of the definition of \mathcal{P}_s . Hence, $p^* \in \mathcal{P}$,

and, from Lemma 2-(ii), $p^* \in \Delta_{\underline{\delta}}$. Then, Lemma 1-(ii) and the above choice of n and r imply $\|z^*\| + \|x^*\| < r_{\underline{\delta}} < n < r$.

From Claims 8-9, and from Remark 4, the collection $((p^*, q^*), (\Sigma_i), (x^*, z))$ belongs to $\Pi_{\underline{\delta}} \times \Gamma \times \Pi_{i=1}^m B_i(\Sigma_i, p^*, q^*)$ and satisfies Conditions (b) and (c) of Definition 2 of equilibrium. Assume, by contraposition, that it does not meet Condition (a) of that Definition 2. Then (replacing \tilde{z}_i by its projection on $Z_i^{o\perp}$ if needed), there exist $i \in I$ and $(x_i, \tilde{z}_i) \in B_i(\Sigma_i, p^*, q^*) \cap P_i^{\Sigma_i}(x_i^*) \times Z_i^{o\perp}$. From Assumption A4, from the relation $\|z^*\| + \|x^*\| < n < r$, and from the convexity of $B_i(\Sigma_i, p^*, q^*)$ and $P_i^{\Sigma_i}(x_i^*) \times Z_i^{o\perp}$, the better strategy (x_i, \tilde{z}_i) may be chosen sufficiently close to (x_i^*, z_i^*) so that $(x_i, \tilde{z}_i) \in B_i(p^*, q^*) \cap P_i(x_i^*) \times Z_i$, which contradicts Claims 11-12. Hence, $((p^*, q^*), (\Sigma_i), (x^*, z))$ also meets Condition (a) of Definition 2, i.e., is an equilibrium of $\mathcal{E}[V, (S_i)]$. From Claim 13, that equilibrium satisfies the price conditions of Theorem 1. Finally, Theorem 1 holds, since $(\Sigma_i) \in \mathcal{S}[V, (S_i)]$ was set arbitrary. \square

Appendix

We prove the Lemmas of Section 3, using the notations of pages 7-8, namely:

$$\begin{aligned} \Delta &:= \{p \in (\mathcal{R}^L)^{S'} : \|p[s]\| \leq 1, \forall s \in S', p^l[\bar{s}] \geq \varepsilon, \forall (l, \bar{s}) \in \{1, \dots, L\} \times S \setminus \underline{\mathbf{S}}\}; \\ \Delta^* &:= \{p \in \Delta : \|p[s]\| = 1, \forall s \in \underline{\mathbf{S}} := \cap_i S_i\}; \\ \Delta_\delta &:= \{p \in \Delta : p[s] \cdot e \geq \delta, \forall s \in S\} \neq \emptyset, \text{ for each } \delta \in]0, \varepsilon[; \\ Q &:= \{q \in Z^{o\perp} : \|q\| \leq 1\}; \\ \Pi &:= \Delta \times Q; \Pi^* := \Delta^* \times Q \text{ and } \Pi_\delta := \Delta_\delta \times Q; \end{aligned}$$

and, for each $n \geq 1$, and every $(i, (p, q)) \in I \times \Pi$,

$$\begin{aligned} \overline{B}_i(p, q) &:= \{(x, z) \in X(\Sigma_i) \times Z_i^{o\perp} : (p_\square[x - e_i])[\Sigma_i'] \leq W(\Sigma_i, p, q)z + 1[\Sigma_i']\}; \\ \overline{B}_i^n(p, q) &:= \{(x, z) \in \overline{B}_i(p, q) : \|z\| \leq n\}; \\ \overline{\mathcal{A}}(p, q) &:= \{[(x_i, z_i)] \in \Pi_{i=1}^m \overline{B}_i(p, q) : (x_i) \in \mathcal{A}(\Sigma_i), \sum_{i=1}^m z_i \in Z^o\}; \\ \text{and } \overline{\mathcal{A}}^n(p, q) &:= \{[(x_i, z_i)] \in \Pi_{i=1}^m \overline{B}_i^n(p, q) : (x_i) \in \mathcal{A}(\Sigma_i), \sum_{i=1}^m z_i \in Z^o\}. \end{aligned}$$

Lemma 1 *Given the structure $(\Sigma_i) \in \mathcal{S}[V, (S_i)]$ and the above definitions of the sets $\Pi, \Pi_\delta, \overline{\mathcal{A}}(p, q), \overline{\mathcal{A}}^n(p, q)$ (for $((p, q), \delta) \in \Pi \times]0, \varepsilon[$ and $n \geq 1$), the following Assertions hold:*

- (i) $\forall n \geq 1, \exists r_n > 0 : [(p, q) \in \Pi \ \& \ (x, z) \in \overline{\mathcal{A}}^n(p, q)] \implies [\|x\| + \|z\| < r_n];$
- (ii) $\forall \delta \in]0, \varepsilon[, \exists r_\delta > 0 : [(p, q) \in \Pi_\delta \ \& \ (x, z) \in \overline{\mathcal{A}}(p, q)] \implies [\|x\| + \|z\| < r_\delta].$

Proof Let $\delta \in]0, \varepsilon[$ and $n \geq 1$ be given.

- First, there exists r^1 such that: $x := (x_i) \in \mathcal{A}(\Sigma_i) \implies \|(x[\underline{\mathbf{S}}'])\| < r^1$.

Indeed, let $\alpha := \max_{(i,s,l) \in I \times S' \times \{1, \dots, L\}} e_i^l[s]$ and $r^1 := 1 + m^2 L(\#S')\alpha$. For each $i \in I$, the consumption set $X(\Sigma_i) \subset (\mathcal{R}_+^L)^{S'}$ is bounded below by zero. Hence, $x := (x_i) \in \mathcal{A}(\Sigma_i)$ implies $\sum_{i=1}^m x_i[\underline{\mathbf{S}}'] = 0$ and $0 \leq x_i[\underline{\mathbf{S}}'] \leq \sum_{j \in I} e_j[\underline{\mathbf{S}}']$, therefore, $\|x_i[\underline{\mathbf{S}}']\| \leq mL(\#S')\alpha$, for each $i \in I$, that is, $\|(x[\underline{\mathbf{S}}'])\| < r^1$.

- Second, we show that, for every $M > 0$, there exists r^M , such that:
 $[(p, q) \in \Pi, (x, z) \in \overline{\mathcal{A}}(p, q) \text{ and } \|z\| < M] \implies [\|x\| < r^M]$, which suffices to prove Lemma 1-(i).

Indeed, denoting $\beta := \max_{(s,j) \in S \times \{1, \dots, J\}} |v_j[s]|$, we let $M > 0$, $(p, q) \in \Pi$ and $(x, z) := [(x_i, z_i)] \in \overline{\mathcal{A}}(p, q)$ be given, such that $\|z\| < M$. The definition of Δ_δ yields $0 \leq x_i^l[s] \leq \alpha + [1 + \beta M J]/\varepsilon$, for each pair $(i, l) \in I \times \{1, \dots, L\}$ and each state $s \in \Sigma_i \setminus \Sigma$. Let $\gamma^M := \alpha + [1 + \beta M J]/\varepsilon$ and $r^M := r^1 + Lm(\#S)\gamma^M$. Then, from above, $\|x\| < r^M$.

- Finally, there exists $M_\delta > 0$, such that:
 $[(p, q) \in \Pi_\delta \text{ and } (x, z) := [(x_i, z_i)] \in \overline{\mathcal{A}}(p, q)] \implies [\|z\| < M_\delta]$.

Indeed, assume, by contraposition, that, for every positive integer k , there exist $(p^k, q^k) \in \Pi_\delta$ and $(x^k, z^k) := [(x_i^k, z_i^k)] \in \overline{\mathcal{A}}(p^k, q^k)$, such that $\|z^k\| > k$. Each set $\overline{\mathcal{A}}(p^k, q^k)$ (for $k > 0$) is closed and convex and contains (x^k, z^k) and $[(e_i, 0)]$. By construction, $(x'^k, z'^k) := [(x_i'^k, z_i'^k)] := [\frac{1}{\|z^k\|}(x_i^k + [\|z^k\| - 1]e_i, z_i^k)]$ satisfies $(x'^k, z'^k) \in \overline{\mathcal{A}}(p^k, q^k)$ and $\|z'^k\| = 1$, for each $k \geq 1$. From above, the sequences $((x'^k, z'^k))_{k \geq 1}$ and $((p^k, q^k))_{k \geq 1}$ are bounded in some Euclidean space and in a compact set, and may be assumed to converge, say, towards (x', z') and $(p', q') \in \Pi_\delta$. Since the scalar product is continuous and the correspondence $(p, q) \mapsto \overline{\mathcal{A}}(p, q)$ is closed, $\|z'\| = 1$ and $(x', z') \in \overline{\mathcal{A}}(p', q')$.

Given $K > 1$, we let, for every $k > K$, $x_K^k := \frac{K}{\|z^k\|}x^k + (1 - \frac{K}{\|z^k\|})(e_i)$. By the same token, for $k > K$, $(x_K^k, Kz'^k) \in \overline{\mathcal{A}}(p^k, q^k)$ and the sequence $((x_K^k, Kz'^k))_{k > K}$ may be assumed to converge to some $(x_K, z_K) \in \overline{\mathcal{A}}(p', q')$, such that $z_K = Kz'$ (for the same z' and (p', q') as above). Hence, for every $K > 1$, there exists $(x_K, z_K) \in \overline{\mathcal{A}}(p', q')$, such that $z_K = Kz'$, with $\|z'\| = 1$.

All consumption sets being bounded below, this yields $V(\Sigma_i, p')z'_i \geq 0$, for every $i \in I$, whereas $(x', z') \in \overline{\mathcal{A}}(p', q')$ implies $\bar{z}' := \sum_{i=1}^m z'_i \in Z^o$. Then, the AFAO Condition and the relation $p' \in \Delta_\delta$ yield $z' := (z'_i) \in \Pi_{i=1}^m Z_i^o$, whereas $(x', z') \in \overline{\mathcal{A}}(p', q')$ implies $z' \in \Pi_{i=1}^m Z_i^{o\perp}$. Hence, $z' \in \Pi_{i=1}^m (Z_i^o \cap Z_i^{o\perp}) = \{0\}$, which contradicts the condition $\|z'\| = 1$, and proves the existence of $M_\delta > 0$, such that: $[(p, q) \in \Pi_\delta \text{ and } (x, z) \in \overline{\mathcal{A}}(p, q)] \implies [\|z\| < M_\delta]$.

- From above, Lemma 1-(ii) holds for $r_\delta = M_\delta + r^{M_\delta}$. \square

We recall that, for each $s \in \underline{S}$: $\mathcal{P}_s := \{p_s \in \mathfrak{R}^L : \exists p \in \Delta^* \text{ s.t. } p_s = p[s], \exists i \in I, \exists x := (x_i) \in \mathcal{A}(\Sigma_i) \text{ s.t. } (y_i \in P_i(x) \text{ and } y_i[S' \setminus \{s\}] = x_i[S' \setminus \{s\}]) \text{ imply } (p[s] \cdot y_i[s] \geq p[s] \cdot x_i[s] \geq p[s] \cdot e_i[s])\}$; and that $\mathcal{P} := \{p \in \Delta^* : p[s] \in \mathcal{P}_s, \forall s \in \underline{S}\}$.

Lemma 2 *The following Assertions hold:*

- (i) *for each $s \in \underline{S}$, \mathcal{P}_s is closed, hence, \mathcal{P}_s and \mathcal{P} are compact sets;*
- (ii) *there exists $\underline{\delta} \in]0, \varepsilon[$, such that $\mathcal{P} \subset \Delta_{\underline{\delta}}$.*

Proof

(i) Let $s \in \underline{S}$ and a converging sequence $(p_s^k)_{k \geq 1}$ of elements of \mathcal{P}_s be given. Since Δ^* is closed, there exists $p \in \Delta^*$, s.t. $p_s := \lim p_s^k = p[s]$. Moreover, w.l.o.g., we may assume there exist $i \in I$ and a sequence $(x^k)_{k \geq 1} := ((x_i^k))_{k \geq 1}$ of elements of $\mathcal{A}(\Sigma_i)$, such that, for each $k \geq 1$, (p_s^k, i, x^k) satisfies the conditions of the definition of \mathcal{P}_s . For every $k \geq 1$, the reader will readily check (from the separability of utility functions) that we may take $x_j^k[S \setminus \{s\}] := e_j[S \setminus \{s\}]$, for every $j \in I$. Moreover, from the proof of Lemma 1, $((x_i^k[0], x_i^k[s]))_{k \geq 1}$ is bounded. Thus, we assume at no cost that $(x^k)_{k \geq 1}$ converges, say to $x \in \mathcal{A}(\Sigma_i)$, and that $x_j^k[S \setminus \{s\}] := e_j[S \setminus \{s\}]$, for every $k \geq 1$ and every $j \in I$.⁵

The relations $p_s^k \cdot (x_i^k[s] - e_i[s]) \geq 0$, for every $k \geq 1$, yield, in the limit, $p_s \cdot (x_i[s] - e_i[s]) \geq 0$. We show that (p_s, i, x) satisfies the conditions of the definition of \mathcal{P}_s (hence, $p_s := \lim p_s^k \in \mathcal{P}_s$, i.e., \mathcal{P}_s is closed). Assume, by contraposition, that this is not the case. Then, from above, for every $k \geq 1$, there exists $y_i^k \in P_i(x)$, such that $y_i^k[S \setminus \{s\}] = x_i^k[S \setminus \{s\}] := e_i[S \setminus \{s\}]$, $y_i^k[0] = x_i[0]$ and $p_s \cdot (y_i^k[s] - x_i[s]) < 0$, hence, $v_i(\Sigma_i, s)(x_i[0], y_i^k[s]) > v_i(\Sigma_i, s)(x_i[0], x_i[s])$. Let $k \geq 1$ be given and, for every $k' \geq k$, let $y^{k'} \in X(\Sigma_i)$ be defined by $y^{k'}[S] := y_i^k[S]$ and $y^{k'}[0] := x_i^{k'}[0]$. Then, we let the reader check from the latter inequality and Assumption A3, that for some $K \geq k$, big enough, one has $y^{k'} \in P_i(x^{k'})$, for every $k' \geq K$, which implies, by construction of each $x^{k'}$, $p_s^{k'} \cdot (y^{k'}[s] - x_i^{k'}[s]) = p_s^{k'} \cdot (y_i^k[s] - x_i^{k'}[s]) \geq 0$ and, in the limit ($k' \rightarrow \infty$), $p_s \cdot (y_i^k[s] - x_i[s]) \geq 0$. This contradicts the above inequality $p_s \cdot (y_i^k[s] - x_i[s]) < 0$. Hence, $p_s \in \mathcal{P}_s$ and Lemma 2-(i) holds.

(ii) Let $s \in \underline{S}$ be given. We prove, first, that $p[s] \cdot e > 0$ for every $p \in \mathcal{P}$. Indeed, let $p \in \mathcal{P}$ and $(p[s], i, x) \in \mathcal{P}_s \times I \times \mathcal{A}(\Sigma_i)$ meet the conditions of the definition of \mathcal{P}_s . From Assumption A2, there exists $a_i \in X(\Sigma_i)$ such that, $a_i[S' \setminus \{s\}] := x_i[S' \setminus \{s\}]$, and $p[s] \cdot a_i[s] < p[s] \cdot e_i[s] \leq p[s] \cdot x_i[s]$. Then, for every $n > 1$, we let $x_i^n := (\frac{1}{n}a_i + (1 - \frac{1}{n})x_i) \in X(\Sigma_i)$, which satisfies $p[s] \cdot x_i^n[s] < p[s] \cdot x_i[s]$ by construction. Referring to Assumptions A1-A3 and their notations, there exists $n > 1$, such that $y := (x_i^n + (1 - \frac{1}{n})e_s) \in P_i^{\Sigma_i}(x_i)$, which implies, $p[s] \cdot x_i[s] \leq p[s] \cdot y[s] < p[s] \cdot x_i[s] + (1 - \frac{1}{n})p \cdot e$, since $y[S' \setminus \{s\}] = x_i[S' \setminus \{s\}]$. Hence, $p[s] \cdot e > 0$ and, for all $p_s \in \mathcal{P}_s$, there exists $\delta_{p_s} \in]0, \varepsilon[$, such that $p_s \cdot e > \delta_{p_s}$. The mapping $\varphi_s : \mathcal{P}_s \rightarrow \mathbb{R}_{++}$, defined by $\varphi_s(p_s) := p_s \cdot e$ (for $p_s \in \mathcal{P}_s$) is continuous and attain its minimum for some element \underline{p}_s of the compact set \mathcal{P}_s . The reader will readily check that $\underline{\delta} := \min \delta_{\underline{p}_s}$, for $s \in \underline{S}$, satisfies $\mathcal{P} \subset \Delta_{\underline{\delta}}$. This proves Lemma 2-(ii). \square

⁵The Assumption of separable utilities was only used to prove Lemma 2. With symmetric information, the reader will readily check it is not required (since attainable allocations are bounded) and, moreover, that all paper's proofs remain valid if we use preferences correspondences (instead of utility functions), which are open and convex-valued and satisfy the following condition (replacing Assumption A4): $\forall i \in I, \forall \Sigma \subset S, \forall (\lambda, (x, y)) \in]0, 1[\times X(\Sigma)^2$, $y \in P_i^\Sigma(x) \implies (\lambda y + (1 - \lambda)x) \in P_i^\Sigma(x)$

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